

**(Mis-)handling gauge invariance in the theory
of the quantum Hall effect I:
Unifying action and the $\nu = \frac{1}{2}$ state**

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(October 9, 1998)

Abstract

We propose a unifying theory for both the integral and fractional quantum Hall regimes. This theory reconciles the Finkelstein approach to localization and interaction effects with the topological issues of an instanton vacuum and Chern-Simons gauge theory. We elaborate on the microscopic origins of the effective action and unravel a new symmetry in the problem with Coulomb interactions which we name \mathcal{F} -invariance. This symmetry has a broad range of physical consequences which will be the main topic of future analyses. In the second half of this paper we compute the response of the theory to electromagnetic perturbations at a tree level approximation. This is applicable to the theory of ordinary metals as well as the composite fermion approach to the half integer effect. Fluctuations in the Chern-Simons gauge fields are found to be well behaved only when the theory is \mathcal{F} -invariant.

PACSnumbers 72.10.-d, 73.20.Dx, 73.40.Hm

I. INTRODUCTION

It is well known that the quantum Hall effect exists only due to the presence of random impurities¹. Although one usually prefers to think in terms of the pure incompressible states alone, the random impurities problem in all its generality opens up a Pandora's box of concepts and complex analyses. The integral quantum Hall regime is the simplest and most widely studied example. The advances made in this field have in fact very little to do with the Landau quantization of pure states that one originally starts out from¹. Actually, what is really needed is topological ideas in quantum field theory (instanton vacuum^{2,3}) in order to establish a unifying renormalization theory for the many unrelated experimental phenomena that are observed in the laboratory⁴. Some examples of these phenomena are weak localization in weak magnetic fields and higher Landau levels⁵, the quantization of the Hall conductance in strong magnetic fields¹, the (critical) plateau transitions which occur usually in the Landau band centers^{6,7}, the problem of spin unresolved Landau levels etc. The problem that needs to be addressed is characterized by a rich variety of different cross-over length scales, whereas detailed experiments are often difficult to interpret due to the limited range available in temperature and sample-specific properties (long range versus short range disorder⁴ etc.).

To date, the Pandora's box of the integral quantum Hall regime is the only one that has been opened. For the fractional quantum Hall regime, which is believed to be a strongly correlated phenomenon, the impurity problem has not yet even been formulated! Our theoretical understanding has not progressed beyond that of the popularly studied incompressible pure states as initiated by Laughlin⁸.

Nevertheless, there has been a long standing quest for a unifying theory which would combine the basic effects of disorder and strong correlation into a single renormalization group flow diagram of the conductances⁹. The original attempts made in this direction were solely motivated by the experiments which seemed to indicate that the integral and fractional effects have very similar common features.

In this paper, and others that follow, we will lay the foundation and construct the much sought after unifying theory. However, formulation of this unifying theory heavily relies on advances made in the recent literature on the fractional and integer quantum Hall effect. In particular, we refer to the analysis of localization and interaction made by two of the authors¹⁰, in which Finkelstein's effective sigma model theory¹¹ was extended by including topological effects (instanton vacuum)¹⁰. In this work it was shown that the interacting electron gas shares many of the basic features which were previously found for free electrons, namely asymptotic freedom in two dimensions and non-perturbative renormalization by instantons. These results put the topological concept of an instanton vacuum in an entirely different perspective of many body theory, the consequences of which have yet remained largely unexploited. Secondly, there is the Chern-Simons gauge field approach¹²⁻¹⁸ which implicitly carries out Jain's composite fermion ideas¹⁹ and maps the fractional quantum Hall effect onto the integral effect. We are specifically interested in the fermionic Chern-Simons theory¹⁴, since the basic starting formulation (the fermionic path integral) is the only one suitable for analyzing disorder effects and, in particular, the above mentioned instanton vacuum concept.

The effective action proposed in this paper essentially extends the Finkelstein theory¹¹

to include the topological effects of an instanton vacuum as well as Chern-Simons gauge theory. As one of the principal features of our theory we shall show that most of the presently accumulated knowledge on the quantum Hall regime can be derived from our effective action by considering the extreme limits of *weak* and *strong* coupling only. More specifically, the theory in weak coupling describes the composite fermion approach to the half integral effect. This will be the main subject in the second half of this paper. The theory in strong coupling on the other hand describes the Jain series for fractional quantization of the Hall conductance²⁰ and also provides a microscopic theory of disordered chiral edge states which generalizes and extends the previously introduced Luttinger liquid description for edge states without disorder²¹. Subsequent papers will report the strong coupling effects²².

It is interesting to note that the physics of both weak and strong coupling is essentially a perturbative phenomenon from the instanton vacuum point of view. Nevertheless, our results clearly show that our effective action can be used to establish a much more ambitious theory for the quantum Hall effect. It is possible to address and investigate the consequences of the renormalizability (both perturbative and non-perturbative) of the theory²⁵. Further, this will provide the necessary information on the global phase structure of the quantum transport problem in the presence of random impurities.

In this paper we mainly explain the microscopic origins of the effective action that we propose as the unifying theory for both integral and fractional quantum Hall regimes. The analysis presented is largely based on the insights we have accumulated by extensively studying the free electron renormalization theory of the integral effect. We will therefore refer to this analysis^{3,1} throughout the course of this work. A second important reference which has been critical in motivating this work is the above mentioned renormalization group analysis of localization and interaction effects in the quantum Hall regime¹⁰. During the course of this analysis we became aware of the incomplete nature of Finkelstein's pioneering work on the subject. One of the major complications in Finkelstein's approach is the $U(1)$ electrodynamic gauge invariance of the theory. For example, no transparent and consistent way exists for introducing external vector and scalar potentials and/or Chern-Simons gauge fields. Subsequently this prevents one from using this theory as a microscopic approach for the fractional Hall effect.

In order to construct a unifying theory we start out (Section II) by considering the fermionic path integral in Matsubara frequency representation. We then analyze in a step by step manner both the low energy excitations in the problem (Q -fields) and the $U(1)$ gauge invariance. Upon performing this exercise, we find that the $U(1)$ generators for the gauge fields and those for the Q -fields cover distinctly different sectors in Matsubara frequency space, which naively appear to be completely disconnected. Assuming this to be true, one would not even consider using the effective action approach. However, these two apparently distinct aspects are integrally related to one another via Ward identities (obtained from local $U(N) \times U(N)$ symmetry)²⁵. In order to establish this relationship, several concepts such as 'smallness' of the Q -fields and ' \mathcal{F} -algebra' are introduced. These concepts are absolutely necessary for handling the $U(1)$ gauge invariance of the problem. Additionally, they also elucidate a hidden symmetry in the Finkelstein action which has previously gone unnoticed. The symmetry we unravel has far reaching consequences and plays a critical role in the development of the unifying theory. We term this symmetry as \mathcal{F} -invariance since we will be frequently referring to it in the rest of this work.

One of the important consequences for ordinary metals is that the nature of quantum transport fundamentally changes depending on the length scale being considered. At distances short relative to the Debye screening length, transport is free particle like and conservation laws are governed by Einstein's relation between conduction and diffusion. At large distances, however, the metal is no longer diffusive and the internally generated electric field due to the Coulomb interactions enters into the transport equations. Our theory provides these results by considering the gauge invariant response at the tree level (Section II E).

Finally, in Section IV we include the Chern-Simons statistical gauge fields in the action. As a first step towards describing the fractional Hall regime, we consider the $\nu = 1/2$ state. In this case it is sufficient to work with the statistical gauge fields and external fields in the tree level approximation. Additionally, we compute the contribution of the Chern-Simons gauge fields to the specific heat. For the problem with Coulomb interactions we find that the singularity structure of the theory is not modified. This then demonstrates that the composite fermion approach to the half integral effect is free of infrared trouble. On the other hand, for a system with finite range electron-electron interactions complications do arise. These aspects are further discussed in Section IV C.

We end this paper with a conclusion (Section V).

II. Q-FIELD FORMALISM; THE FERMIONIC PATH INTEGRAL

A. Introduction

We are interested in the disorder average of the logarithm of the grand canonical partition function Z ,

$$Z = \text{tr } e^{\beta(\mu N - H)} \quad (2.1)$$

with β the inverse thermal energy, $\beta = (k_B T)^{-1}$, μ the chemical potential, N the number of electrons and H the total energy of the system. We consider a system of two-dimensional electrons in a random potential $V(\vec{x})$ and a static magnetic field B pointing along the positive z -axis. We work in units where all lengths are expressed in terms of the magnetic length $\ell = \sqrt{2\hbar/eB}$ and where $\hbar=1$, $e=1$. In these units, the electron mass m has the dimension of an inverse energy, while the static magnetic field and the vector potential are dimensionless,

$$m = m_{\text{SI}} \cdot \ell^2 / \hbar^2 \quad ; \quad \vec{A} = \vec{A}_{\text{SI}} \cdot e\ell / \hbar. \quad (2.2)$$

We write the vector potential as $\vec{A}^{\text{st}} + \vec{A}$, where the static part satisfies $\nabla \times \vec{A}^{\text{st}} = B\vec{e}_z$ and \vec{A} represents the quantum fluctuations. In the units chosen above, the magnetic field is normalized to $B=2$. The fluctuations in the scalar potential are denoted by A_0 . In the path integral formulation, the partition function for our system is written in the following way

$$Z = \int \mathcal{D}[\bar{\psi}\psi, A_\mu] e^{S[\bar{\psi}, \psi, A_\mu]} \quad (2.3)$$

$$\begin{aligned} S[\bar{\psi}, \psi, A_\mu] = & \int_0^\beta d\tau \int d^2x \bar{\psi}(\vec{x}, \tau) [-\partial_\tau + iA_\tau(\vec{x}, \tau) + \mu - \mathcal{H}(\vec{x}) - V(\vec{x})] \psi(\vec{x}, \tau) \\ & - \frac{1}{2} \int_0^\beta d\tau \int d^2x d^2x' \bar{\psi}(\vec{x}, \tau) \psi(\vec{x}, \tau) U_0(\vec{x}, \vec{x}') \bar{\psi}(\vec{x}', \tau) \psi(\vec{x}', \tau). \end{aligned} \quad (2.4)$$

Here, the ψ and $\bar{\psi}$ are Grassmann variables defined on the imaginary time interval $\tau \in [0, \beta]$, with the fermionic antiperiodicity condition $\psi(\vec{x}, \beta) = -\psi(\vec{x}, 0)$. The A_μ are ordinary integration variables with the bosonic boundary condition $A_\mu(\vec{x}, \beta) = A_\mu(\vec{x}, 0)$. The $U_0(\vec{x}, \vec{x}')$ is the Coulomb interaction and \mathcal{H} is a differential operator acting to the left and to the right

$$\mathcal{H} = \frac{1}{2m} \overleftarrow{\pi} \cdot \vec{\pi}, \quad (2.5)$$

where π is the covariant derivative,

$$\vec{\pi} = -i\vec{\nabla} - \vec{A}^{\text{st}} - \vec{A}; \quad \overleftarrow{\pi} = i\overleftarrow{\nabla} - \vec{A}^{\text{st}} - \vec{A}. \quad (2.6)$$

The Coulomb term is quartic in the fields ψ . We get rid of this quartic form by performing a Hubbard-Stratonovich transformation, introducing an extra path integration over a bosonic field $\lambda(\vec{x}, \tau)$, the ‘plasmon field’,

$$\begin{aligned} \exp -\frac{1}{2} \int d\tau d^2x d^2x' \bar{\psi}\psi(\vec{x}) U_0(\vec{x}, \vec{x}') \bar{\psi}\psi(\vec{x}') &\propto \\ \int \mathcal{D}[\lambda] \exp \left[-\frac{1}{2} \int d\tau d^2x d^2x' \lambda(\vec{x}) U_0^{-1}(\vec{x}, \vec{x}') \lambda(\vec{x}') + i \int d\tau d^2x \lambda \bar{\psi}\psi \right]. \end{aligned} \quad (2.7)$$

Here U_0^{-1} stands for the matrix inverse of U_0 . In order to find the disorder average $\overline{\ln Z}$ we use the well known *replica trick*^{3,1}. In the path integral formalism this amounts to the introduction of replicated fields $\bar{\psi}^\alpha, \psi^\alpha, \lambda^\alpha, A_\mu^\alpha$ with $\alpha = 1, \dots, N_r$. The quantities μ, ρ, V and A_μ^{st} are identical in all replicas. The replicated partition function is given by

$$\begin{aligned} Z = \int \prod_{\gamma=1}^{N_r} \mathcal{D}[\bar{\psi}^\gamma \psi^\gamma, \lambda^\gamma, A_\mu^\gamma] \exp \sum_{\alpha=1}^{N_r} \int_0^\beta d\tau \left[\int d^2x \bar{\psi}^\alpha [-\partial_\tau + iA_\tau^\alpha + \mu - \mathcal{H}^\alpha - V] \psi^\alpha \right. \\ \left. - \frac{1}{2} \int d^2x d^2x' \lambda^\alpha(\vec{x}) U_0^{-1}(\vec{x}, \vec{x}') \lambda^\alpha(\vec{x}') + i \int d^2x \lambda^\alpha \bar{\psi}^\alpha \psi^\alpha \right]. \end{aligned} \quad (2.8)$$

As a next step we perform a Fourier transform from imaginary time τ to Matsubara frequencies. Since fermionic fields are antiperiodic on the interval $[0, \beta]$, while bosonic fields are periodic, the allowed frequencies for $\psi, \bar{\psi}$ and A_τ, λ are, respectively

$$\omega_n = \frac{2\pi}{\beta} (n + \frac{1}{2}) \quad (\text{fermionic}) \quad ; \quad \nu_n = \frac{2\pi}{\beta} n \quad (\text{bosonic}). \quad (2.9)$$

with n integer. We define the Fourier transformed fields by

$$\begin{aligned} \psi^\alpha(\tau) = \sum_{n=-\infty}^{\infty} \psi_n^\alpha e^{-i\omega_n \tau} \quad ; \quad \bar{\psi}^\alpha(\tau) = \sum_{n=-\infty}^{\infty} \bar{\psi}_n^\alpha e^{+i\omega_n \tau} \\ \lambda^\alpha(\tau) = \sum_{n=-\infty}^{\infty} \lambda_n^\alpha e^{-i\nu_n \tau} \quad ; \quad A_\mu^\alpha(\tau) = \sum_{n=-\infty}^{\infty} (A_\mu)_n^\alpha e^{-i\nu_n \tau} \end{aligned} \quad (2.10)$$

which results in the following form of the action

$$S = \beta \int d^2x \psi^\dagger (i\omega + i\hat{A}_\tau + \hat{\mu} + i\hat{\lambda} - \hat{\mathcal{H}} - \hat{V}) \psi \quad (2.11)$$

$$- \frac{\beta}{2} \int d^2x d^2x' \lambda^\dagger(\vec{x}) U_0^{-1}(\vec{x}, \vec{x}') \lambda(\vec{x}'). \quad (2.12)$$

FIGURES

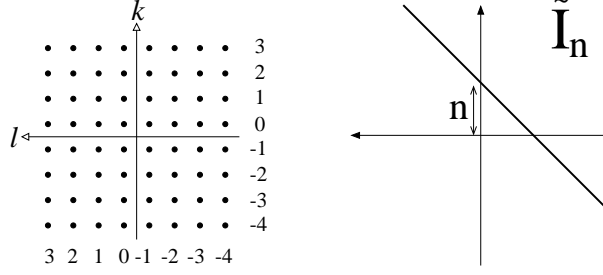


FIG. 1. Our way of picturing a matrix $[\cdot \cdot \cdot]_{kl}$ in frequency space, and the structure of \tilde{I}_n ($n > 0$).

Here we have used matrix notation for combined replica and frequency indices,

$$\psi^\dagger(\cdot \cdot \cdot) \psi = \sum_{nm, \alpha\beta} \bar{\psi}_n^\alpha(\cdot \cdot \cdot)_{nm}^{\alpha\beta} \psi_m^\beta. \quad (2.13)$$

The ω is a unity matrix in replica space, while in Matsubara space it is a diagonal matrix containing the frequencies ω_n

$$(\omega)_{nm}^{\alpha\beta} = \omega_n \delta^{\alpha\beta} \delta_{nm}. \quad (2.14)$$

The ‘hatted’ quantities are defined according to

$$\hat{z} = \sum_{n, \alpha} z_n^\alpha \tilde{I}_n^\alpha \quad \text{with} \quad (\tilde{I}_n^\alpha)_{kl}^{\beta\gamma} = \delta^{\alpha\beta} \delta^{\alpha\gamma} \delta_{k-l, n}. \quad (2.15)$$

The matrix \tilde{I}_n^α is the unit matrix in the α ’th replica space, while in Matsubara space it is zero everywhere except on the n ’th diagonal, where it is 1. The \tilde{I}_n^α -matrices are extremely important, because they will turn out to be the generators of the electromagnetic $U(1)$ transformations. But before we elaborate on this, let us first take the disorder average of the replicated partition sum, in analogy with what has been done in the free particle formalism. This is done using a Gaussian distribution for the random potential $V(\vec{x})$,

$$\begin{aligned} \overline{Z} &= \int \mathcal{D}[V] P[V] Z \\ P[V] &\propto \exp\left(-\frac{1}{2g} \int d^2x V^2\right). \end{aligned} \quad (2.16)$$

This integration leads to a quartic term in the action of the form $(\psi^\dagger \psi)^2$, which can be decoupled by means of a Hubbard-Stratonovich transformation, introducing hermitian matrix field variables $\tilde{Q}_{nm}^{\alpha\beta}(\vec{x})$. The partition function now becomes

$$Z = \int \mathcal{D}[\bar{\psi}\psi, \tilde{Q}, \lambda, A_\mu] e^{S[\bar{\psi}, \psi, \tilde{Q}, \lambda, A_\mu]} \quad (2.17)$$

$$\begin{aligned} S[\bar{\psi}, \psi, \tilde{Q}, \lambda, A_\mu] &= -\frac{1}{2g} \text{Tr} \tilde{Q}^2 + \beta \int d^2x \psi^\dagger [i\omega + i\hat{A}_\tau + \hat{\mu} - \hat{\mathcal{H}} + i\hat{\lambda} + i\tilde{Q}] \psi \\ &\quad - \frac{\beta}{2} \int d^2x d^2x' \lambda^\dagger(\vec{x}) \lambda(\vec{x}') U_0^{-1}(\vec{x}, \vec{x}') \end{aligned} \quad (2.18)$$

where the notation Tr stands for a trace over combined replica and Matsubara indices as well as spatial integration $\int d^2x$. Notice that the only difference with the previously studied free particle case³ is that we work with a Matsubara frequency label, rather than with advanced and retarded components alone.

B. Gauge invariance; tunneling density of states

A generic local $U(1)$ gauge transformation on the fermion fields and the electromagnetic potentials has the form

$$\begin{aligned}\psi^\alpha(\vec{x}, \tau) &\rightarrow e^{i\chi^\alpha(\vec{x}, \tau)}\psi^\alpha(\vec{x}, \tau) \quad ; \quad \bar{\psi}^\alpha(\vec{x}, \tau) \rightarrow e^{-i\chi^\alpha(\vec{x}, \tau)}\bar{\psi}^\alpha(\vec{x}, \tau) \\ A_\mu^\alpha(\vec{x}, \tau) &\rightarrow A_\mu^\alpha(\vec{x}, \tau) + \partial_\mu\chi^\alpha(\vec{x}, \tau)\end{aligned}\tag{2.19}$$

with χ^α real-valued functions periodic in τ . In frequency notation this gauge transformation is written as a unitary matrix acting on the vector ψ

$$\psi \rightarrow e^{i\hat{\chi}}\psi \quad ; \quad \psi^\dagger \rightarrow \psi^\dagger e^{-i\hat{\chi}}\tag{2.20}$$

$$\hat{A}_i \rightarrow \hat{A}_i + \partial_i\hat{\chi} \quad ; \quad (A_\tau)_n^\alpha \rightarrow (A_\tau)_n^\alpha - i\nu_n\chi_n^\alpha.\tag{2.21}$$

From (2.20) it is clear that the \tilde{I} -matrices are the generators of gauge transformations. From their multiplication

$$\tilde{I}_n^\alpha \tilde{I}_m^\beta = \delta^{\alpha\beta} \tilde{I}_{n+m}^\alpha\tag{2.22}$$

it is readily seen that they span an abelian algebra, and that a gauge transformation indeed acts in every replica channel separately, as seen in (2.19). The \tilde{Q} transforms according to

$$\tilde{Q} \rightarrow e^{i\hat{\chi}}\tilde{Q}e^{-i\hat{\chi}}.\tag{2.23}$$

The gauge invariance of the action (2.18) is easily checked: First of all, the plasmon field λ and the combinations $\psi^\dagger\tilde{Q}\psi$ and $\psi^\dagger\psi$ are invariant. Secondly, the fact that the \tilde{I} commute leads to $e^{-i\hat{\chi}}\nabla e^{i\hat{\chi}} = i\nabla\hat{\chi}$, from which it follows that the term $\psi^\dagger\hat{\mathcal{H}}\psi$ is also invariant. Finally, using the following commutation relation,

$$[\tilde{I}_n^\alpha, \omega] = -\nu_n\tilde{I}_n^\alpha,\tag{2.24}$$

in combination with the transformation rule (2.21) for A_τ , we find that the term $\psi^\dagger(i\omega + i\hat{A}_\tau)\psi$ is also invariant. In the partition function (2.17), the integration over fermion fields can be performed, yielding an effective action for the variables \tilde{Q} , A_μ and λ ,

$$\begin{aligned}S[\tilde{Q}, \lambda, A_\mu] &= -\frac{1}{2g}\text{Tr } \tilde{Q}^2 + \text{Tr } \ln[i\omega + i\hat{A}_\tau + \hat{\mu} - \hat{\mathcal{H}} + i\hat{\lambda} + i\tilde{Q}] \\ &\quad -\frac{\beta}{2}\int d^2x d^2x' \lambda^\dagger(\vec{x})\lambda(\vec{x}')U_0^{-1}(\vec{x}, \vec{x}').\end{aligned}\tag{2.25}$$

The gauge invariance of this effective action is again evident. We only have to rewrite the gauge transformed second term $\text{Tr } \ln[\dots + ie^{i\hat{\chi}}\tilde{Q}e^{-i\hat{\chi}}]$ into $\text{Tr } \ln[e^{-i\hat{\chi}}(\dots)e^{i\hat{\chi}} + i\tilde{Q}]$ and repeat the arguments given above.

We end this section with an expression for the one particle Green's function $G(\tau_2 - \tau_1)$ which enters the tunneling density of states,

$$G(\tau_2 - \tau_1) = \langle \bar{\psi}^\alpha(\vec{x}, \tau_2)\psi^\alpha(\vec{x}, \tau_1) \rangle.\tag{2.26}$$

In terms of the \tilde{Q} -variable this expression is written as

$$\langle \tilde{Q}^{\alpha\alpha}(\tau_1, \tau_2) \rangle = \sum_{nm} e^{-i\omega_n \tau_1} e^{i\omega_m \tau_2} \langle \tilde{Q}_{nm}^{\alpha\alpha} \rangle. \quad (2.27)$$

The reader may verify that under a gauge transformation, the Green's function transforms as

$$e^{i[\chi^\alpha(\tau_2) - \chi^\alpha(\tau_1)]} \langle \tilde{Q}^{\alpha\alpha}(\tau_1, \tau_2) \rangle \quad (2.28)$$

as it should.

C. Truncation of frequency space

1. 'Large' and 'small' components

We proceed as in the free particle analysis and split the \tilde{Q} matrix variable into 'transverse' and 'longitudinal' components,

$$\tilde{Q} = T^{-1} P T \quad P = P^\dagger \quad T \in SU(2N'). \quad (2.29)$$

Here, P has only block-diagonal components in frequency space (i.e. $P_{nm}^{\alpha\beta}$ is nonzero only for $\omega_n \cdot \omega_m > 0$) and T is a unitary rotation. $2N'$ is the size of the Matsubara space times the number of replica channels, and represents the size of the \tilde{Q} -matrix.

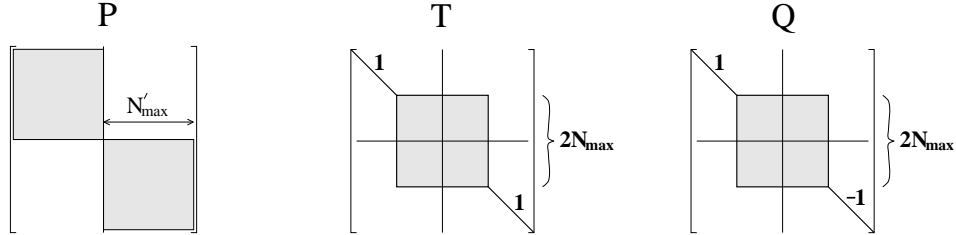


FIG. 2. Sketch of the matrices P , T and Q in truncated frequency space ($N_{\max} \ll N'_{\max}$).

This change of variables (2.29) is motivated by the saddlepoint structure of the theory (2.25) in the absence of the fields A_μ, λ and at zero temperature (i.e. $\omega_n \rightarrow 0$). This saddlepoint can be written as

$$\tilde{Q}_{\text{sp}} \propto T^{-1} \Lambda T \quad ; \quad \Lambda_{nm}^{\alpha\beta} = \delta^{\alpha\beta} \delta_{nm} \text{sgn}(\omega_n), \quad (2.30)$$

indicating that the longitudinal fluctuations P are the 'massive' components of the theory whereas the T -matrix fields are the lowest energy excitations (Goldstone modes) in the problem. The manner in which (2.29) is going to be used is illustrated in Fig. 2: we impose on the T -rotations a cutoff N_{\max} in Matsubara frequency space, such that $1 \ll N_{\max} \ll N'_{\max}$. It is important to keep in mind that working with a finite N_{\max} is just a calculational device which will enable us to derive an effective action for the lowest energy excitations T by formally integrating out the massive components P (the latter can be done explicitly by employing saddlepoint methods). Once an effective action for the T -fields has been obtained,

we have to find some procedure by which the cutoff N_{\max} can be sent to infinity. The main problem is to ensure that such a procedure retains the electrodynamic $U(1)$ gauge invariance of the theory. We will return to this problem at the end of section II D. A more formal justification of the ‘smallness’ concept is postponed until section II E where we introduce vector and scalar potentials in the effective action.

2. \mathcal{F} -invariance

In order to be able to carry through the concept of ‘small’ T -rotations in a ‘large’ Matsubara frequency space, we shall need to perform specific algebraic manipulations which (sometimes) will be referred to by the name of ‘ \mathcal{F} -algebra’. To illustrate the meaning of this algebra we shall next derive the effective action for the fields T in the presence of Coulomb interactions, but without scalar and vector potentials. The effective action is defined by

$$e^{S_{\text{eff}}[T]} \propto \int \mathcal{D}[P, \lambda] \exp \left(-\frac{1}{2g} \text{Tr} P^2 - \frac{\beta}{2} \int d^2x d^2x' \lambda^\dagger(\vec{x}) U_0^{-1}(\vec{x}, \vec{x}') \lambda(\vec{x}') \right. \\ \left. + \text{Tr} \ln[i\omega + \hat{\mu} + i\hat{\lambda} - \hat{\mathcal{H}} + iT^{-1}PT] \right). \quad (2.31)$$

In two spatial dimensions, the Coulomb interaction is infinitely ranged. The Fourier transform is given by

$$U_0(\vec{q}) \propto \int d^2x \frac{e^{-i\vec{q} \cdot \vec{x}}}{|\vec{x}|} \propto |\vec{q}|^{-1}. \quad (2.32)$$

From general symmetry considerations one can impose two important conditions on the actual form of $S_{\text{eff}}[T]$.

1. The only local variable on which $S_{\text{eff}}[T]$ can depend and which is consistent with the symmetries of the problem is precisely of the form of \tilde{Q}_{sp} (2.30),

$$Q = T^{-1} \Lambda T. \quad (2.33)$$

Here the matrices are all acting in ‘large’ frequency space of size $2N'_{\max} \times 2N'_{\max}$. The T -rotations are effectively ‘small’ ($2N_{\max} \times 2N_{\max}$ with $N_{\max} \ll N'_{\max}$) as depicted in Fig. 2.

2. The effective action must be invariant under global (i.e. spatially independent) ‘ W -rotations’:

$$S_{\text{eff}}[Q] = S_{\text{eff}}[W_0 Q W_0^{-1}] \quad \text{with} \quad W_0 = \exp i \sum_{n, \alpha} \chi_n^\alpha I_n^\alpha, \quad (2.34)$$

where the matrix I_n^α stands for \tilde{I}_n^α truncated to size $2N'_{\max} \times 2N'_{\max}$.

The statement (2.34), which is exact in the limit $N'_{\max} \rightarrow \infty$, can easily be derived from (2.31) by using the invariance of the $\text{Tr} \ln$ under unitary transformations. This amounts to a spatially independent shift of the plasmon field λ inside the $\text{Tr} \ln$ according to

$$\lambda_n^\alpha(\vec{x}) \rightarrow \lambda_n^\alpha(\vec{x}) + (\partial_\tau \chi)_n^\alpha. \quad (2.35)$$

This shift can be absorbed in a redefinition of λ provided that the interaction U_0 is infinitely ranged (i.e. $U_0^{-1}(\vec{q}) \rightarrow 0$ as $|\vec{q}| \rightarrow 0$), as considered here. The ‘ \mathcal{F} -invariance’ of (2.34) plays a very special role in the problem. Notice that (2.34) actually stands for a global $U(1)$ gauge transformation and is directly related to the statement of macroscopic charge conservation. The far-reaching consequences of this statement were understood first by Finkelstein.

3. Effective action

We will proceed by presenting S_{eff} in an \mathcal{F} -invariant manner as follows²⁷

$$S_{\text{eff}}[Q] = S_\sigma[Q] + S_F[Q] + S_U[Q]. \quad (2.36)$$

The first term, S_σ , is precisely the nonlinear σ model action in the presence of the instanton term,

$$S_\sigma[Q] = -\frac{1}{8}\sigma_{xx}^0 \text{Tr} (\nabla Q)^2 + \frac{1}{8}\sigma_{xy}^0 \text{Tr} \varepsilon_{ij} Q \partial_i Q \partial_j Q \quad (2.37)$$

where σ_{xx}^0 and σ_{xy}^0 denote the (mean field) conductances in units e^2/h . The second term, S_F , can be written as

$$S_F[Q] = \frac{1}{2}z\frac{\pi}{\beta}\sum'_{n\alpha} \text{Tr} [I_n^\alpha, Q][I_{-n}^\alpha, Q]. \quad (2.38)$$

The quantity z is the ‘singlet interaction amplitude’ and it defines the temperature scale. The prime on the summation over n indicates a restriction on the frequency range, $n \in \{-N'_{\text{max}}, \dots, N'_{\text{max}}-1\}$. The last term in (2.36) contains the Coulomb interaction U_0 explicitly and can be written as

$$S_U[Q] = -\frac{\pi}{\beta}\sum_{n\alpha} \int d^2x d^2x' [\text{tr} I_n^\alpha Q(\vec{x})] U^{-1}(\vec{x} - \vec{x}') [\text{tr} I_{-n}^\alpha Q(\vec{x}')]. \quad (2.39)$$

In momentum space U^{-1} is given by

$$U^{-1}(p) = \int \frac{d^2r}{2\pi} U^{-1}(\vec{r}) e^{-i\vec{p}\cdot\vec{r}} = \frac{\pi}{2} \cdot \frac{1}{\rho^{-1} + U_0(p)} \quad (2.40)$$

where $\rho = \partial n / \partial \mu$ is the *thermodynamic density of states*.

4. Examples of \mathcal{F} -algebra

We stress again that from now onward, all matrix manipulations are done in truncated $(2N'_{\text{max}} \times 2N'_{\text{max}})$ Matsubara frequency space. The truncated I-matrices obviously no longer obey the simple $U(1)$ algebra (2.22), but instead

$$(I_n^\alpha I_m^\beta)_{kl}^{\mu\nu} = (\tilde{I}_n^\alpha \tilde{I}_m^\beta)_{kl}^{\mu\nu} g_{l+m} \quad ; \quad [I_n^\alpha, I_m^\beta]_{kl}^{\mu\nu} = \delta^{\alpha\beta\mu\nu} \delta_{k-l, m+n} (g_{l+m} - g_{l+n}) \quad (2.41)$$

where $\delta^{\alpha\beta\mu\nu}$ means that all replica indices have to be the same, and g_i is a step function equal to one if $i \in \{-N'_{\max}, \dots, N'_{\max}-1\}$ and zero otherwise. Consequently, the W_0 in (2.34) stands for a more complicated unitary matrix of size $2N'_{\max} \times 2N'_{\max}$. Nevertheless, by making use of elementary but subtle algebra one can show that the procedure with an arbitrary ‘large’ cutoff correctly describes the low energy sector and correctly retains the electrodynamic gauge invariance of the theory at low frequencies. We proceed by listing some of the important subtleties of the \mathcal{F} -algebra.

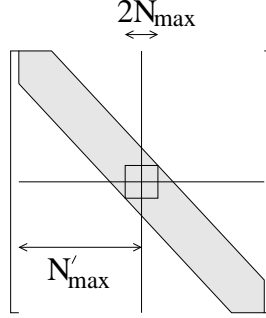


FIG. 3. The summation interval $n \in \{-2N_{\max}+1, \dots, 2N_{\max}-1\}$, indicated by the shaded area.

The definition of (2.38), which involves the restricted frequency sum, is particularly delicate. It can be written in a more familiar form by first writing

$$Q = \Lambda + \delta Q. \quad (2.42)$$

Written out explicitly, (2.38) now becomes

$$S_F[Q] = z \frac{\pi}{\beta} \int d^2x \left[\sum_{\alpha} \sum_{klmn} \delta Q_{kl}^{\alpha\alpha} \delta Q_{mn}^{\alpha\alpha} \delta_{k-l, n-m} + 4 \text{tr } \eta \delta Q \right] + \text{const.} \quad (2.43)$$

which is the result originally obtained by Finkelstein. Here η is a matrix representation of the Matsubara frequencies,

$$\eta_{nm}^{\alpha\beta} = n \delta^{\alpha\beta} \delta_{nm}. \quad (2.44)$$

The constant in (2.43) is proportional to $\text{tr } \eta \Lambda = N_r \sum_{n=-L}^{L-1} |n|$, where the cutoff L depends on the exact definition of the prime in (2.38). Mostly, we will not be interested in the exact value of L , and a prime on a summation simply means a restriction to small frequencies. Using the definition of the prime under (2.38), Eq. (2.43) can be re-expressed in terms of Q as follows

$$S_F[Q] = z \frac{\pi}{\beta} \int d^2x \left[\sum_{\alpha n} (\text{tr } I_n^{\alpha} Q) (\text{tr } I_{-n}^{\alpha} Q) + 4 \text{tr } \eta Q - 6 \text{tr } \eta \Lambda \right]. \quad (2.45)$$

Notice that the bilinear forms in Q in (2.38) and (2.45) differ by a frequency term $\text{tr } \eta Q$ and a constant. Within Finkelstein’s formulation of the problem (2.43), the very special relative coefficient “4” between the first (singlet interaction) and second (frequency) term arises from

the macroscopic conservation laws in a very indirect and deep manner. The advantage of the present formalism is (amongst other things) the simple algebraic interpretation of the result which can be obtained from the symmetries of the problem. Moreover, the constant appearing in (2.45) has a very special significance for physical quantities such as the specific heat. This aspect of the problem will be further discussed in subsequent work²⁶.

5. General remarks

For a general understanding of the result (2.36) we next discuss the various pieces separately. First, by putting the temperature β^{-1} equal to zero we obtain the same result S_σ as in the free electron theory. The ‘bare’ parameters σ_{xx}^0 and σ_{xy}^0 are generally modified by the electron-electron interactions. The modifications are of a Fermi-liquid type and in the limit of strong magnetic fields the results depend on the ratio of disorder energy Γ_0 (the width of the Landau band) and the typical Coulomb energy E_0 ($= U_0(\ell)$, where ℓ is the magnetic length).

The most important part next is S_F . Quite unlike what one naively might expect, the presence of S_F alters the ultraviolet singularity structure of the free electron theory S_σ altogether²⁶. This peculiar aspect of the problem indicates that the electronic system with Coulomb interactions has a behavior fundamentally different from that with finite range interactions or free electrons.

Next we briefly elaborate on the significance of the Coulomb term S_U (2.39,2.40) which is usually ignored in renormalization group analyses, since it really stands for a higher dimensional operator (notice that $U^{-1}(p) \propto |p|$ in the small momentum limit). The importance of this term, however, can not be overemphasized. First, we mention that in the large momentum limit we can substitute $\frac{\pi}{2}\rho$ for z in (2.38)²⁷. In this limit the sum of S_U and S_F does not contain the term quadratic in $(\text{tr } IQ)$ and (2.36) reduces to the effective action for *free* particles. This means that the full theory of (2.36-2.39) is appropriately interpreted in terms of a cross-over phenomenon between free electron behaviour at short distances (or high temperature β^{-1}) and an interaction dominated behaviour which appears at larger distances (or lower temperatures β^{-1}) only.

Secondly, the complete form of (2.39) and (2.40) unravels important information on the nature of quantum transport even for ordinary metals. This will be the main subject of section III, where we compute at a tree level the complete momentum and frequency dependent response of the theory to electromagnetic perturbations.

Finally, we mention that although (2.39) is naively irrelevant from the weak coupling renormalization point of view, it nevertheless dominates the quantum transport problem in the strong coupling (insulating) phase which is characterized by strong interaction effects such as the appearance of the Coulomb gap in the (quasiparticle) density of states. This will be the main subject of a subsequent paper, where we embark on the renormalization group behaviour of the theory²⁶.

D. Gauge invariance in truncated frequency space

In this section we wish to show that S_{eff} (2.36) is \mathcal{F} -invariant, i.e. it satisfies the requirement stated in (2.34). We will proceed by giving the results for an arbitrary spatially *dependent* gauge or W-transformation from which the statement of \mathcal{F} -invariance follows trivially. Assuming that W approaches unity at spatial infinity, we obtain

$$S_\sigma[WQW^{-1}] = -\frac{1}{8}\sigma_{xx}^0 \text{Tr} (\vec{d}Q)^2 + \frac{1}{8}\sigma_{xy}^0 \text{Tr} \varepsilon_{ij} Q \partial_i Q \partial_j Q \quad (2.46)$$

where

$$\vec{d}Q = [\nabla + i\nabla\hat{\chi}, Q]. \quad (2.47)$$

Furthermore we have[†]

$$S_F[WQW^{-1}] = S_F[Q] \quad (2.48)$$

$$\begin{aligned} S_U[WQW^{-1}] = & -\frac{\pi}{\beta} \sum_{\alpha n} \int d^2x d^2x' \left[\text{tr} I_n^\alpha Q(\vec{x}) + \frac{\beta}{\pi} (\partial_\tau \chi)_{-n}^\alpha(\vec{x}) \right] U^{-1}(\vec{x}, \vec{x}') \\ & \times \left[\text{tr} I_{-n}^\alpha Q(\vec{x}') + \frac{\beta}{\pi} (\partial_\tau \chi)_n^\alpha(\vec{x}') \right]. \end{aligned} \quad (2.49)$$

For completeness we give the results for the W-transformations of $\text{tr} I_n^\alpha Q$ and $\text{tr} \eta Q$:

$$\text{tr} I_n^\alpha WQW^{-1} = \text{tr} I_n^\alpha Q + \frac{\beta}{\pi} (\partial_\tau \chi)_{-n}^\alpha \quad (2.50)$$

$$\text{tr} \eta WQW^{-1} = \text{tr} \eta Q - \frac{\beta}{2\pi} \text{tr} Q \widehat{\partial_\tau \chi} - \left(\frac{\beta}{2\pi}\right)^2 (\partial_\tau \chi)^\dagger (\partial_\tau \chi). \quad (2.51)$$

The remarkable aspect of these results (2.46-2.51) is that the W-rotation on the Q does not contribute beyond the lowest few orders in a power series expansion in the I-matrices! What is more, the arbitrary cutoff N'_{max} does not enter these final results and can be safely taken to infinity. Next, from (2.51) we see that within the Q -field formalism the frequency matrix ω does not transform simply according to the linear rule $\omega \rightarrow \omega - \widehat{\partial_\tau \chi}$ as one would naively expect. The consistency of the \mathcal{F} -algebra demands that terms quadratic in the gauge field χ are being generated such that the Finkelstein part of the action (S_F) as a whole remains gauge invariant. The results of the \mathcal{F} -algebra are therefore somewhat counterintuitive.

In summary we can say that electrodynamic gauge transformations can be incorporated in the Q -field theory for localization and interaction effects. For this purpose we introduced the ‘smallness’ concept for the Q -fields, whereas the W or electrodynamic gauge transformations are considered to be ‘large’. In the next section (II E) we will build upon these findings and present a formal justification of our cutoff procedure in Matsubara frequency space. In practice this means that the coupling between the ‘large’ W -rotations and the ‘small’ Q -matrix fields as discussed in this section is the *only consistent way* of carrying through electrodynamic gauge transformations in the effective action formalism. The stringent requirements on the cutoff procedure do not, however, provide an answer to the fundamental question of $U(1)$ gauge *invariance*. More specifically, since the Q and WQW^{-1} do not (by construction) belong to the same manifold, we generally can not absorb the W -rotation into the measure of the Q -integration and prove the gauge invariance in this way. The general idea behind this approach, however, is that gauge invariance is only obtained after the cutoff

N_{\max} in the effective action is sent to infinity. This way of handling the $U(1)$ gauge invariance is completely new and special care should therefore be taken. The proof of gauge invariance of the Finkelstein theory ultimately relies on the results of explicit, laborious calculations, both perturbative²⁶ and non-perturbative¹⁰. This, then, puts extra weight on statements of renormalizability and we will embark on this problem in subsequent papers.

From now onward we are going to treat the W -rotations and \mathcal{F} -invariance as a good symmetry of the problem, keeping in mind that the limit $N_{\max} \rightarrow \infty$ is always taken in the end.

E. External fields

One may next employ the results of the previous section and extend the theory by including vector and scalar potentials A_μ . This could be done in such a way that the resulting action is invariant under the transformation $Q \rightarrow e^{i\hat{\chi}} Q e^{-i\hat{\chi}}$, $A_\mu^\alpha \rightarrow A_\mu^\alpha + \partial_\mu \chi^\alpha$. Such a procedure, however, does not imply anything for the topological piece of the action S_σ , which couples to external fields in a more complicated fashion. In anticipation of a detailed analysis of disordered edge currents²² we report the following results.

$$S_\sigma \rightarrow -\frac{1}{8}\sigma_{xx}^0 \text{Tr} ([\vec{D}, Q])^2 + \frac{1}{8}\sigma_{xy}^0 \varepsilon_{ij} \text{Tr} Q [D_i, Q] [D_j, Q] - \frac{\beta}{8\pi^2 \rho} (\sigma_{xy}^{II})^2 \int d^2x B^\dagger B \quad (2.52)$$

$$S_U \rightarrow -\frac{\pi}{\beta} \sum_{n\alpha} \int \frac{d^2q}{(2\pi)^2} U^{-1}(\vec{q}) \left[\text{tr} I_n^\alpha Q(-\vec{q}) - \frac{\beta}{\pi} (\tilde{A}_\tau)_n^\alpha(-\vec{q}) \right] \left[\text{tr} I_{-n}^\alpha Q(\vec{q}) - \frac{\beta}{\pi} (\tilde{A}_\tau)_n^\alpha(\vec{q}) \right] \quad (2.53)$$

where we have defined

$$\tilde{A}_\tau = A_\tau - \frac{i}{2\pi\rho} \sigma_{xy}^{II} B. \quad (2.54)$$

The terms containing $\sigma_{xy}^{II} \propto \partial n / \partial B^{5,1}$ are the result of the diamagnetic edge currents in the problem, which give rise to extra contributions. We stress that the complete microscopic result of (2.52) clearly demonstrates the theoretical subtleties of the effective action procedure which can not be taken for granted. In addition to this, we mention that (2.52) and (2.53) really stand for extremely nontrivial statements made on the low-energy, long wavelength excitations of the theory. In order to see this, we consider the theory (2.52, 2.53) at a classical level, i.e. we put $Q = \Lambda$. The results now represent an effective action for the external fields A_μ which contains the *same* microscopic parameters σ_{ij}^0 etc. as those appearing in $S_{\text{eff}}[Q]$ (2.36-2.39) without external fields. This result is truly remarkable if one realizes that the effective actions $S_{\text{eff}}[A_\mu, Q = \Lambda]$ and $S_{\text{eff}}[A_\mu = 0, Q]$ follow from fundamentally different expansion procedures applied to the original theory (2.25-2.29). In appendix A we elaborate further on this point and show that the different expansion procedures are in fact related by Ward identities. These Ward identities are not only crucially important in the microscopic derivation of the general result (2.52, 2.53), they also provide a formal justification of the ‘smallness’ concept. In appendix B we give a simple example and show how the theory (2.36-2.39) can be obtained in this way.

III. RESPONSE AT TREE LEVEL

A. Perturbative expansion

It is straightforward to check that the Q -field theory at a classical level (putting $Q=\Lambda$) does not provide a gauge invariant response to the external fields A_μ . In order to obtain a $U(1)$ invariant result, one has to work with the propagators of the Q -field fluctuations. A $U(1)$ invariant result at a so-called tree level is obtained by taking the Q -field fluctuations to lowest order into account.

The most effective way to proceed is to first make use of a W - or gauge transformation such that the A_τ in (2.53) is absorbed into the vector potential \vec{A} . It is easy to check that under such a W -rotation the fields transform according to

$$\begin{aligned}\vec{A}_n^\alpha &\rightarrow z_n^\alpha = \vec{A}_n^\alpha + \frac{\nabla(A_\tau)_n^\alpha}{i\nu_n} \\ A_\tau &\rightarrow 0.\end{aligned}\tag{3.1}$$

It is obviously advantageous to deal directly with the gauge invariant quantity $\vec{z}_n^\alpha = i\vec{E}_n^\alpha/\nu_n$, where \vec{E} is the electric field ($\partial_\tau \vec{A} - \nabla A_\tau$). In order to define a perturbative expansion in the Q -field we write

$$Q = \begin{pmatrix} \sqrt{1 - qq^\dagger} & q \\ q^\dagger & -\sqrt{1 - q^\dagger q} \end{pmatrix}\tag{3.2}$$

where the matrices q, q^\dagger are taken as independent field variables. We use the following convention for the Matsubara indices: the quantities n_1, n_3, \dots with odd subscripts run over non-negative integers, such that the corresponding fermionic frequencies ω_{n_i} are positive. By the same token, the n_2, n_4, \dots run over negative integers and the corresponding ω_{n_i} are all negative. The action can be written as a series in powers of the fluctuation fields q, q^\dagger . The propagators of the Gaussian theory are given by

$$\begin{aligned}\langle q_{n_1 n_2}^{\alpha\beta}(p) [q^\dagger]_{n_4 n_3}^{\gamma\delta}(-p') \rangle &= \frac{4}{\sigma_{xx}^0} \delta^{\alpha\delta} \delta^{\beta\gamma} \delta(p - p') \delta_{n_{12}, n_{34}} D_p(n_{12}) \times \\ &\times \left\{ \delta_{n_1 n_3} + \delta^{\alpha\beta} \kappa^2 [z - U^{-1}(p)] D_p^c(n_{12}) \right\}\end{aligned}\tag{3.3}$$

$$\begin{aligned}D_p(m) &= [p^2 + \kappa^2 m z]^{-1} \quad ; \quad D_p^c(m) = [p^2 + \kappa^2 m U^{-1}(p)]^{-1} \\ \kappa^2 &= \frac{8\pi}{\beta \sigma_{xx}^0} \quad ; \quad n_{12} = n_1 - n_2.\end{aligned}$$

We obtain the following result for the response at tree level

$$S[A_\mu] = -\sigma_{xx}^0 \sum_{\alpha, n > 0} \int \frac{d^2 p}{(2\pi)^2} n \overline{(z_i)_n^\alpha(p)} \left[\delta_{ij} - \frac{p_i p_j}{p^2 + \kappa^2 n U^{-1}(p)} \right] (z_j)_n^\alpha(p)\tag{3.4}$$

where, for simplicity, we have put $\sigma_{xy}^0 = 0$ for the moment. (The bar-notation indicates complex conjugation.) The theory (3.4) provides important physical information on the

process of quantum transport. In order to show this we write for the electron density n (using $\tau = it$)

$$-\beta n_m^\alpha(p) = \frac{\delta S[A_\mu]}{\delta(A_\tau)_m^\alpha(-p)} = -\frac{\beta\sigma_{xx}^0}{2\pi} p_i \left[\delta_{ij} - \frac{p_i p_j}{p^2 + \kappa^2 m U^{-1}(p)} \right] (z_j)_m^\alpha(p) \quad (3.5)$$

which can be written as

$$[\nu_m + \frac{1}{4}\sigma_{xx}^0 p^2 U(p)] n_m^\alpha = \frac{\sigma_{xx}^0}{\beta} m \vec{p} \cdot \vec{z}_m^\alpha = i \vec{p} \cdot (\vec{j}_{\text{ext}})_m^\alpha. \quad (3.6)$$

We have obtained a current density on the r.h.s. by using $\vec{z}_m^\alpha = i \vec{E}_m^\alpha / \nu_m$ and $\vec{j}_{\text{ext}} = \frac{\sigma_{xx}^0}{2\pi} \vec{E}$.

Eq. (3.6) can be rewritten in the form

$$[\nu_m + p^2 D_{xx}^0] (n_c)_m^\alpha + i \frac{\sigma_{xx}^0}{2\pi} \vec{p} \cdot [\vec{E} - i \vec{p} U_0 n_c]_m^\alpha = 0, \quad (3.7)$$

where $n_c = -n$ is the charge density and D_{xx}^0 the diffusion constant, equal to $\sigma_{xx}^0 / 2\pi\rho$ by the Einstein relation. In spacetime notation (3.7) reads

$$\partial_t n_c + \nabla \cdot (\vec{j}_{\text{diff}} + \vec{j}_c) = 0 \quad (3.8)$$

and expresses the well known result from the theory of metals²⁸ with $\vec{j}_{\text{diff}} = -D_{xx}^0 \nabla n_c$ being the diffusive current and $\vec{j}_c = \frac{\sigma_{xx}^0}{2\pi} \vec{E}_{\text{tot}}$ the conductivity current generated by the total electric field inside the system:

$$\vec{E}_{\text{tot}} = \vec{E} - \nabla \int d^2 x' U_0(x - x') n_c(x'). \quad (3.9)$$

Notice that in the limit of low momenta (or high frequencies) the \vec{j}_{diff} in (3.8) can be neglected and the system only responds to the sum of externally applied and internally generated electric fields. The instantaneous Coulomb potential apparently wins over the much slower diffusive processes in this case. In a separate paper²⁷ we address the problem of quantum corrections to the semiclassical theory (3.8).

B. Including magnetic fields

The general result (2.52) describes interesting edge dynamics in case strong magnetic fields are present²². In the remainder of this paper, however, we will limit ourselves to the problem of weak magnetic fields, in which case the σ_{xy}^{II} term can be neglected¹ and edge effects become immaterial. The topological piece of (2.52) can then be written as

$$\frac{1}{4} \text{Tr} \varepsilon^{ij} Q [D_i, Q] [D_j, Q] = \frac{1}{4} \text{Tr} \varepsilon^{ij} Q \partial_i Q \partial_j Q + i \text{Tr} Q \nabla \times \hat{z} + \int d^2 x \sum_{\alpha, n} n \vec{z}_n^\alpha \times \vec{z}_{-n}^\alpha. \quad (3.10)$$

This leads to the following gauge invariant response

$$S[A_\mu] = - \sum_{\alpha, n > 0} \int \frac{d^2 p}{(2\pi)^2} n \overline{(z_i)_n^\alpha(p)} \left[\sigma_{xx}^0 \delta_{ij} + \sigma_{xy}^0 \varepsilon_{ij} \right] (z_j)_n^\alpha(p) \quad (3.11)$$

$$+ \frac{1}{\sigma_{xx}^0} \sum_{\alpha, n > 0} \int \frac{d^2 p}{(2\pi)^2} n \left[\sigma_{xx}^0 \vec{p} \cdot \vec{z}_n^\alpha + \sigma_{xy}^0 \vec{p} \times \vec{z}_n^\alpha \right]^* D_p^c(n) \left[\sigma_{xx}^0 \vec{p} \cdot \vec{z}_n^\alpha - \sigma_{xy}^0 \vec{p} \times \vec{z}_n^\alpha \right].$$

If we now repeat the calculation of the electron density in section III using the action (3.11), we find that the results (3.6)-(3.8) still hold, with one modification: The ‘external’ current \vec{j}_{ext} and the internally generated current \vec{j}_c now also include a Hall current,

$$j_i = \frac{\sigma_{xx}^0}{2\pi} E_i + \frac{\sigma_{xy}^0}{2\pi} \varepsilon_{ij} E_j. \quad (3.12)$$

(The modification of \vec{j}_c is not apparent in the calculations, however, since $\nabla \cdot \vec{j}_c^{\text{Hall}} \propto \varepsilon_{ij} q_i q_j U(q) n(q) = 0$.) For convenience later on, we write the result (3.11) in terms of new variables Φ, Ψ

$$z_i = \partial_i \Phi + \varepsilon_{ij} \partial_j \Psi \quad (3.13)$$

$$S[\Phi, \Psi] = -\sigma_{xx}^0 \sum_{\alpha, n > 0} \int \frac{d^2 p}{(2\pi)^2} n p^2 (\Phi^*, \Psi^*) M \begin{pmatrix} \Phi \\ \Psi \end{pmatrix}. \quad (3.14)$$

The 2×2 matrix M is given by

$$M = \begin{bmatrix} \mathcal{G} & -\omega_c \tau \mathcal{G} \\ \omega_c \tau \mathcal{G} & 1 + (\omega_c \tau)^2 (1 - \mathcal{G}) \end{bmatrix} \quad \text{with} \quad \mathcal{G} = 1 - p^2 D_p^c(n) \quad (3.15)$$

where we have made use of the semiclassical notation

$$\sigma_{xx}^0 = \frac{\sigma_0}{1 + (\omega_c \tau)^2} \quad ; \quad \sigma_{xy}^0 = \omega_c \tau \sigma_{xx}^0. \quad (3.16)$$

IV. CHERN-SIMONS GAUGE FIELDS

A. Introducing CS gauge fields

The results of the previous sections are easily extended to include statistical gauge fields and the Chern-Simons action, leading to the composite fermion description of the half-integer effect in the quantum Hall regime. The action (2.4) now becomes

$$S[\bar{\psi}, \psi, A_\mu] \rightarrow S[\bar{\psi}, \psi, A_\mu + a_\mu] + \frac{i}{4\pi} \frac{1}{2p} \int a \wedge da \quad (4.1)$$

where we have used the shorthand notation

$$\int a \wedge da = \int_0^\beta d\tau \int d^2 x \varepsilon^{\mu\nu\lambda} a_\mu \partial_\nu a_\lambda. \quad (4.2)$$

Equation (4.1) describes the coupling of an even number ($2p$) of flux quanta to each electron, but it leaves the physical amplitudes of the theory formally unchanged. This flux binding transformation has been exploited at many places elsewhere and it leads to the composite fermion description of the qHe. The action (4.1) can be directly translated into Q -field theory by writing $A_\mu \rightarrow A_\mu + a_\mu$ with one important subtlety, namely that the zero-frequency

components of a_μ obviously commute with the T -rotations and hence belong to the underlying theory with the P matrix field. These zero-frequency components can be treated in mean-field theory. Writing $b = \nabla \times \vec{a}$, the mean field equation for the Chern-Simons magnetic field is

$$b = -2p n(B + b) \quad (4.3)$$

where the composite fermion density $n(B + b)$ is defined by

$$n(B + b) = L^{-2} \text{Tr} \left[\overline{i\omega + \mu + i\hat{\lambda} - \mathcal{H}(\vec{A}^{\text{cl}} + \vec{a}_0) + iP} \right]^{-1}. \quad (4.4)$$

Here L^2 is the size of the system and the bar denotes the average with respect to the action (2.31) with $T=1$. Since we know that the density of a half-filled Landau band is given by $B/2\Phi_0$ (with Φ_0 the flux quantum h/e), it immediately follows from (4.3) that near half filling the Chern-Simons field b must cancel the external field B almost completely, provided $p \approx 1$. Hence the composite fermion problem turns into a weak magnetic field problem which can be handled with the methodology of this paper. This leads to an extension of the actions (2.52) and (2.53),

$$S \rightarrow S_{\text{cs}}[a] + S_\sigma[A + a] + S_U[A + a] \quad (4.5)$$

where the a stands for all but the zero-frequency components of the CS field, and $S_{\text{cs}}[a]$ is defined as the $\int a \wedge da$ term in (4.1). It is understood that now $\sigma_{ij}^0 = \sigma_{ij}^0(B + b)$, for which the semiclassical form (3.16) is a good approximation. Equation (4.5) can be written in the form (3.14) as follows

$$S \rightarrow S[\Phi + \varphi, \Psi + \psi] + S_{\text{cs}}[\varphi, \psi] \quad (4.6)$$

$$S_{\text{cs}}[\varphi, \psi] = -\sigma_{xx}^0 \sum_{\alpha, n > 0} \int \frac{d^2 q}{(2\pi)^2} n q^2 (\varphi^*, \psi^*) M_{\text{cs}} \begin{pmatrix} \varphi \\ \psi \end{pmatrix} \quad (4.7)$$

$$M_{\text{cs}} = \frac{1}{2p\sigma_{xx}^0} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

where we have used the transverse gauge $a_i = \varepsilon_{ij} \partial_j \psi$, $a_\tau = -\partial_\tau \varphi$.

B. Mapping of conductances

The conductances of the composite fermion system can be obtained by integrating over the CS field a . For instance, working with the action (3.11) with $A \rightarrow A + a$, one can put $|q| \rightarrow 0$ first; it suffices to take the first two terms only. This leads to a mapping of the composite fermion conductances σ_{ij}^0 to measurable quantities σ_{ij} ,

$$\sigma_{xx} = \frac{\sigma_{xx}^0}{(2p\sigma_{xx}^0)^2 + (2p\sigma_{xy}^0 + 1)^2} \quad ; \quad \sigma_{xy} = \frac{1}{2p} \left[1 - \frac{2p\sigma_{xy}^0 + 1}{(2p\sigma_{xx}^0)^2 + (2p\sigma_{xy}^0 + 1)^2} \right]. \quad (4.8)$$

This ‘mapping’ is known to be a realization of $\text{Sl}(2, \mathbb{Z})$. It becomes exact in the limit $\sigma_{xx}^0 \rightarrow \infty$, which is the weak coupling case considered here. Equation (4.8) applies also to the case where $\sigma_{xx}^0 \rightarrow 0$ but with a modified definition of σ_{xy}^0 which is now integrally quantized.

It is reasonable to assume that (4.8) gives a good overall description provided one is not too close to the critical plateau transition. In this case one expects the conductances to be broadly distributed. This then complicates the relation between average and measured conductances, and (4.8) is likely to be affected by the higher order response terms which have been neglected in (3.11)¹⁷.

We can also integrate out the CS field a_μ working with the full action (3.11) instead of only the first two terms. The resulting action for A_μ has the exact form of (3.11), with σ_{ij}^0 replaced by the mapped σ_{ij} (4.8).

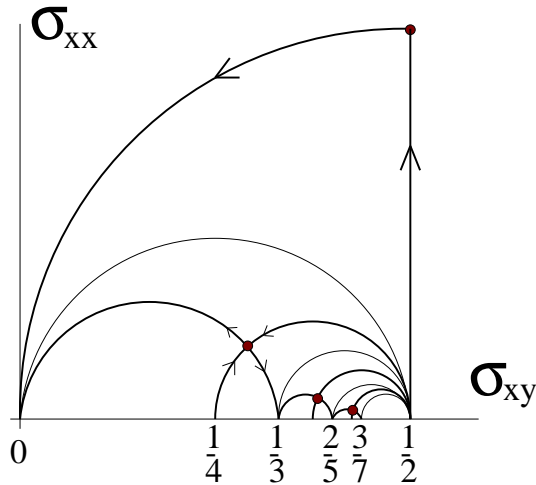


FIG. 4. Unifying RG diagram for integral and fractional quantum Hall states. After Ref 23.

It is important to remark that the $Sl(2,Z)$ mapping (4.8) is neither unique nor universal, but that it depends on microscopic details of the system such as disorder. For example, the CS gauge fields require a different treatment in a theory with slowly varying potential fluctuations, resulting in a different mapping between integer and fractional regimes.²⁴

C. Internal energy; specific heat

In order to decide whether the fluctuations in the CS gauge fields are well-behaved, we next compute the free energy and extract from it the specific heat. We employ (4.6) as well as (3.14), (3.15). For $p=1$ we write

$$\det[M + M_{cs}] = \mathcal{G} \left[1 + (\omega_c \tau)^2 + \frac{\omega_c \tau}{\sigma_{xx}^0} \right] + (2\sigma_{xx}^0)^{-2}. \quad (4.9)$$

The contribution to the free energy can be written as

$$\delta F = \sum_{\alpha, n > 0} \int \frac{d^2 q}{(2\pi)^2} \ln \left\{ \mathcal{G} \left[1 + (\omega_c \tau)^2 + \frac{\omega_c \tau}{\sigma_{xx}^0} \right] + (2\sigma_{xx}^0)^{-2} \right\}. \quad (4.10)$$

In particular, we consider the derivative with respect to temperature

$$\begin{aligned}
\frac{\partial \delta F}{\partial \ln T} &= \sum_{\alpha, n > 0} \int \frac{d^2 q}{(2\pi)^2} \frac{1 + (\omega_c \tau)^2 + \omega_c \tau / \sigma_{xx}^0}{\mathcal{G}[1 + (\omega_c \tau)^2 + \frac{\omega_c \tau}{\sigma_{xx}^0}] + (2\sigma_{xx}^0)^{-2}} \left[\frac{q^2}{q^2 + \kappa^2 n U^{-1}} - \left(\frac{q^2}{q^2 + \kappa^2 n U^{-1}} \right)^2 \right] \\
&= \sum_{\alpha, n > 0} \int \frac{d^2 q}{(2\pi)^2} \frac{\kappa^2 n [1 + (\omega_c \tau)^2 + \omega_c \tau / \sigma_{xx}^0]}{\kappa^2 n [1 + (\omega_c \tau)^2 + \frac{\omega_c \tau}{\sigma_{xx}^0}] + (2\sigma_{xx}^0)^{-2} (\kappa^2 n + q^2 U)} \cdot \frac{q^2}{q^2 + \kappa^2 n U^{-1}}. \quad (4.11)
\end{aligned}$$

This expression is well-behaved in the infrared and for small ν_n it can be written in the form

$$\frac{\partial \delta F}{\partial \ln T} = \sum_{n > 0} \nu_n \rho(\nu_n) \quad (4.12)$$

where $\rho(\omega)$ is of order $\omega \ln \omega$ for small ω . However, for free particles or short-ranged interactions the insertion of CS gauge fields leads to singular contributions, since by putting U constant one finds $\rho(\omega) \approx |\ln \omega|$ for small ω . This implies that the CS fields lead to a singular quasiparticle density of states

$$\rho_{qp}(\varepsilon) = \rho(i\varepsilon) + \rho(-i\varepsilon) \quad (4.13)$$

entering the expression for the specific heat²⁶. The exact meaning of the Chern-Simons gauge field procedure is not clear in this case. The results nevertheless demonstrate the fundamental significance of \mathcal{F} -invariance in the problem, possibly indicating that a new saddlepoint should be found for the finite range interaction problem. This, then, shows the importance of the Coulomb interactions.

V. CONCLUSION

In this paper we have embarked on the subject of electrodynamic gauge invariance in the Finkelstein approach to localization and interaction phenomena. We have found a new symmetry in the problem (\mathcal{F} -invariance) which has fundamental implications in setting up a unifying theory for the quantum Hall effect. The proposed unifying theory reconciles Finkelstein's effective action with the topological concepts of an instanton vacuum and Chern-Simons gauge theory. Forthcoming analyses will further investigate this theory. The second half of this paper has been devoted to the consequences of \mathcal{F} -invariance for ordinary metals as well as the composite fermion approach to the half-integer effect.

Future work on this subject will include the tunneling density of states which will have direct significance for recent experiments.

ACKNOWLEDGEMENTS

This research was supported in part by INTAS (grant #96-0580).

Appendix A: Justification of ‘smallness’

In order to demonstrate the validity of the ‘smallness’ concept (section II C, Fig 2) let us proceed from the most difficult part of the action, (2.31),

$$\text{Tr} \ln[i\omega + i\hat{A}_\tau + \hat{\mu} - \hat{\mathcal{H}} + i\hat{\lambda} + iT^{-1}PT] \quad (\text{A1})$$

and reflect on the possibility of constructing an effective action $S_{\text{eff}}[A_\mu, T]$ which contains the two distinctly different sets of field variables A_μ and T simultaneously. Notice that the A_μ are ‘large’ matrices, and the problem is specified to the question as to why only ‘small’ T -rotations are the relevant low energy excitations. For this purpose we consider the effective actions $S_{\text{eff}}[A_\mu, T=1]$ and $S_{\text{eff}}[A_\mu=0, T]$ separately. More explicitly, write

$$e^{S_{\text{eff}}[A_\mu, 1]} = \int \mathcal{D}[P, \lambda] I[P] \exp \left\{ -\frac{\beta}{2} \int \lambda^\dagger U_0^{-1} \lambda + \text{Tr} \ln[i\omega + i\hat{A}_\tau + \hat{\mu} - \hat{\mathcal{H}} + i\hat{\lambda} + iP] \right\} \quad (\text{A2})$$

$$e^{S_{\text{eff}}[0, T]} = \int \mathcal{D}[P, \lambda] I[P] \exp \left\{ -\frac{\beta}{2} \int \lambda^\dagger U_0^{-1} \lambda + \text{Tr} \ln[i\omega + \hat{\mu} - \hat{\mathcal{H}}_0 + i\hat{\lambda} + iT^{-1}PT] \right\}. \quad (\text{A3})$$

The $S_{\text{eff}}[A_\mu, T=1]$ is formally obtained by expanding the $\text{Tr} \ln$ to lowest order in powers of the large matrices A_μ and this means that complicated infinite sums over frequencies will have to be performed. The situation for $S_{\text{eff}}[A_\mu=0, T]$ is quite different and one has to follow the procedure of³ in order to formally express this action in terms of the ‘small’ variable Q to lowest orders in a derivative and temperature expansion.

However, since one is usually interested in the limit of small momenta, frequencies and temperatures, only a finite number of terms in S_{eff} needs to be considered in both cases. The coefficients are microscopic parameters which are generally given as complex correlations defined by the underlying theory with plasmon (λ) and disorder (P) variables alone. These coefficients (coupling constants) of $S_{\text{eff}}[A_\mu, 1]$ and $S_{\text{eff}}[0, T]$ are related to one another by gauge invariance, as will be shown next, and this then provides the starting point for constructing a complete action $S_{\text{eff}}[A_\mu, T]$, which is done by ‘matching’ the known results for both pieces. The details of how to do this are described, in part, in this paper (Appendix B) and in a forthcoming paper on the Luttinger liquid behaviour of disordered edge states in the qHe.

To establish a relation between (A2) and (A3) we start out by taking a pure gauge for the A_μ in (A2), i.e. $A_\mu = \partial_\mu \varphi$, and a ‘large’ matrix field $T = e^{-i\hat{\varphi}}$ in (A3). Writing

$$\hat{\varphi} = \sum_{\alpha, |n| < M} \varphi_n^\alpha \mathbf{I}_n^\alpha \quad (\text{A4})$$

(A2) and (A3) certainly stand for one and the same thing. Next, we make this statement useful by showing that the large rotation $T = e^{-i\hat{\varphi}}$ can be replaced by an equivalent rotation (t) which is small. For this purpose we write

$$\hat{\varphi} = \hat{\varphi}_t + \hat{\varphi}_l \quad (\text{A5})$$

where $\hat{\varphi}_l$ is block diagonal (nonzero only in the $++$ and $--$ Matsubara blocks) and $\hat{\varphi}_t$ is block off-diagonal (nonzero in the $+-$ and $-+$ blocks). Now write

$$T = e^{-i\kappa\hat{\varphi}} = e^{-i\kappa\hat{\varphi}_l} t(\kappa) \quad (\text{A6})$$

where the parameter κ formally serves as an expansion parameter. The $t(\kappa)$ can be written as a series in powers of κ as follows:

$$t(\kappa) = \exp \sum_{n=1}^{\infty} \frac{i\kappa^n}{n!} \hat{x}_n \quad (\text{A7})$$

where

$$\hat{x}_1 = \hat{\varphi}_t \quad \hat{x}_2 = [\hat{\varphi}_t, \hat{\varphi}_l] \quad \hat{x}_3 = [[\hat{\varphi}_t, \hat{\varphi}_l], \hat{\varphi}_l] \quad \dots \quad (\text{A8})$$

The important point is that the \hat{x}_n are all block off-diagonal matrices and their ‘size’ in frequency space increases linearly in n . It serves our purpose to truncate the series beyond small orders in κ such that $t(\kappa)$ satisfies the condition of ‘smallness’. The statement (A6) now effectively turns into a separation of large components $e^{-i\kappa\hat{\varphi}_l}$ and small components $t(\kappa)$. The large components can be absorbed into a redefinition of the P -field which leads to the statement

$$T^{-1}PT = t^{-1}(\kappa)Pt(\kappa) \quad (\text{A9})$$

or, equivalently,

$$S_{\text{eff}}[\kappa\partial_\mu\varphi, 1] = S_{\text{eff}}[0, t(\kappa)]. \quad (\text{A10})$$

This procedure can be extended as follows: Suppose we have found $S_{\text{eff}}[A_\mu, T]$ from a ‘matching’ procedure as mentioned above. A useful check upon this result is obtained by a generalization of (A10),

$$S_{\text{eff}}[A_\mu + \kappa\partial_\mu\varphi, t(\kappa)] = S_{\text{eff}}[A_\mu, 1]. \quad (\text{A11})$$

Appendix B

In order to give an example of the matching procedure (Appendix A), we derive an effective action in the plasmon field λ and the matrix field variable T . Define $S_{\text{eff}}[\lambda, T = 1]$ and $S_{\text{eff}}[\lambda = 0, T]$ as follows,

$$e^{S_{\text{eff}}[\lambda, 1]} = \int \mathcal{D}P \, I[P] \exp\left\{-\frac{\beta}{2} \int \lambda^\dagger U_0^{-1} \lambda + \text{Tr} \ln[i\omega + \mu - \hat{\mathcal{H}}_0 + i\hat{\lambda} + iP]\right\} \quad (\text{B1})$$

$$e^{S_{\text{eff}}[0, T]} = \int \mathcal{D}P \, I[P] \exp\{\text{Tr} \ln[i\omega + \mu - \hat{\mathcal{H}}_0 + iT^{-1}PT]\}. \quad (\text{B2})$$

The idea is to construct $S_{\text{eff}}[\lambda, T]$ from a detailed knowledge of Eqs. (B1) and (B2). Notice that (B2) is precisely the free particle problem. Eq (B2) is evaluated by writing the $\text{Tr} \ln[]$ as

$$\text{Tr} \ln[iT\omega T^{-1} + \mu - T\hat{\mathcal{H}}_0 T^{-1} + iP] = \text{Tr} \ln[\mu - \hat{\mathcal{H}}_0 + iP + X], \quad (\text{B3})$$

where $X = iT\omega T^{-1} - T[\hat{\mathcal{H}}_0, T^{-1}]$ is a small parameter. An expansion in powers of X leads to a systematic expansion of $S_{\text{eff}}[\lambda = 0, T]$ in powers of the gradient and temperature. The result can be expressed in the field variable Q as follows

$$S_{\text{eff}}[0, T] = S_\sigma[Q] + \frac{2\pi}{\beta} \pi \rho_0 \text{Tr } \eta Q + \dots \quad (\text{B4})$$

where ρ_0 is the free particle density of states, equal to $dn/d\mu$ in this case, which can be written as

$$\rho_0 = -\frac{1}{2\pi i} \left\langle G_{n_1 n_1}^{\alpha\alpha}(x, x) - G_{n_2 n_2}^{\alpha\alpha}(x, x) \right\rangle_{\text{av}} \quad ; \quad G(x, x') = \langle x | (\mu - \hat{\mathcal{H}}_0 + iP)^{-1} | x' \rangle \quad (\text{B5})$$

where the average is with respect to the theory of (B2) with $T=1$ and $\omega=0$. In (B5) the indices are kept fixed with $n_1 > 0$ and $n_2 < 0$ as usual. Eq. (B5) is identical to the more familiar expression for ρ_0 , as can be seen from the standard rules of replica field theory. More specifically, for quantities like (B5) which involve unmixed averages over the positive and negative blocks of P , one can transform the problem back and trade in the P -integral for the average over the original random potential $V(\vec{x})$. On the other hand, we write (B1) as an expression in powers of λ . The result to lowest order in λ can be written as

$$S_{\text{eff}}[\lambda, 1] = -\frac{\beta}{2} \sum_{n\alpha} \int \lambda_{-n}^\alpha U_0^{-1} \lambda_n^\alpha - \frac{1}{2} \sum_{\alpha\beta} \sum_{n,m} \int \lambda_n^\alpha M_{nm}^{\alpha\beta} \lambda_m^\beta \quad (\text{B6})$$

where

$$M_{nm}^{\alpha\beta}(x, x') = -\text{tr} \left\langle \hat{G}(x, x') I_m^\beta \hat{G}(x', x) I_n^\alpha \right\rangle_{\text{av}} + \left\langle \text{tr} [\hat{G}(x, x) I_m^\beta] \text{tr} [\hat{G}(x', x') I_n^\alpha] \right\rangle_{\text{cum}} \\ \hat{G}(x, x') = \langle x | (i\omega + \mu - \hat{\mathcal{H}}_0 + iP)^{-1} | x' \rangle. \quad (\text{B7})$$

The subscript ‘cum’ stands for the cumulant average with respect to (B1) with $\lambda=0$. Notice the subtle difference in the expansions of (B1) and (B2), in that the ω -matrix is treated differently in (B7) and (B5), leading to different propagators \hat{G} and G , respectively.

The matrix elements $M_{nm}^{\alpha\beta}$ in (B6) can be simplified by making use of the fact that the expectations of \hat{G} (B7) are invariant under unitary transformations. Specifically, (B7) is invariant under the replacement

$$\hat{G}_{nm}^{\alpha\beta}(x, x') = [U^{-1} \hat{G}(x, x') U]^{\alpha\beta}_{nm}, \quad (\text{B8})$$

where U is diagonal in the Matsubara frequency index, $U_{nm}^{\alpha\beta} = \delta_{nm} U_m^{\alpha\beta}$. It is then straightforward to show that $M_{nm}^{\alpha\beta}$ must be of the general form

$$M_{nm}^{\alpha\beta}(x, x') = \delta^{\alpha\beta} \delta_{n+m,0} M_1(x - x', \omega_n) + \delta_{m,0} \delta_{n,0} M_0(x - x'). \quad (\text{B9})$$

The two different terms in (B9) have an entirely different meaning and they are going to be treated quite differently in what follows. First, the quantity M_1 can be expanded in a series expansion in small momenta (gradients) and frequencies. To lowest order we have

$$M_1(x - x', \omega_n) = \beta \rho_0 \delta(x - x') + \dots \quad (\text{B10})$$

Here, ρ_0 can be identified as the exact free particle density of states (B5). The \dots in (B10) stands for all the higher order terms in frequency and derivatives. They become important only when higher dimensional operators in Q (represented by \dots in (B4)) are taken into account.

Next, the zero frequency quantity $M_0(x - x')$ in (B9) can be identified as the ‘mean field’ result for the density fluctuation correlation $\overline{\delta n(x)\delta n(x')}$, where $\delta n(x) = n(x) - \overline{n(x)}$ and the bar denotes the ensemble average. We shall see that full $M_0(x - x')$ (i.e. without momentum expansion) completely decouples from the effective action procedure and is, in fact, immaterial.

The idea then is to find the ‘match’ between the different series expansions (B6–B10) and (B4). Schematically, this is given by

$$S_{\text{eff}}[\lambda, T] = -\frac{\beta}{2} \sum_{n\alpha} \int \lambda_{-n}^\alpha U_0^{-1} \lambda_n^\alpha - \frac{1}{2} \sum_{\alpha\beta} \int \lambda_0^\alpha M_0 \lambda_0^\beta + S_\sigma[Q] \\ + \frac{2\pi}{\beta} \pi \rho_0 \int \left[\text{tr } \eta Q + \frac{\beta}{2\pi} \sum_{n\alpha} \lambda_n^\alpha \text{tr } I_n^\alpha Q - \left(\frac{\beta}{2\pi}\right)^2 \sum_{n\alpha} \lambda_{-n}^\alpha \lambda_n^\alpha \right]. \quad (\text{B11})$$

It can be shown that (B11) satisfies (A11). Eq. (B11) therefore is the desired result. Moreover, comparison with (2.51) shows that (B11) is \mathcal{F} -invariant. Next, by making the appropriate shift

$$\lambda_n^\alpha \rightarrow \lambda_n^\alpha + \frac{\pi}{\beta} \rho_0 (U_0^{-1} + \rho_0)^{-1} \text{tr } I_{-n}^\alpha Q,$$

the final result decouples such that we have

$$S_{\text{eff}}(\lambda, T) = S_{\text{eff}}[\lambda] + S_\sigma[Q] + S_F[Q] + S_U[Q] \quad (\text{B12})$$

where

$$S_{\text{eff}}[\lambda] = -\frac{\beta}{2} \int \lambda^\dagger (U_0^{-1} + \rho_0) \lambda - \frac{1}{2} \sum_{\alpha\beta} \int \lambda_0^\alpha M_0 \lambda_0^\beta \\ S_F[Q] = \frac{\pi^2}{2\beta} \rho_0 \int \left[\sum_{n\alpha} \text{tr } I_n^\alpha Q \text{tr } I_{-n}^\alpha Q + 4 \text{tr } \eta Q \right] \\ S_U[Q] = -\frac{\pi}{\beta} \sum_{n\alpha} \int \text{tr } (I_n^\alpha Q) U^{-1} \text{tr } (I_{-n}^\alpha Q). \quad (\text{B13})$$

This is precisely the form written in (2.36–2.39). The procedure that has taken us from (B1,B2) to (B13) can be systematically extended to include higher orders. This means that terms of higher dimension in (B4) and (B10) as well as higher powers of λ in (B6) can be taken into account. The extended procedure leads to renormalization (in the Fermi liquid sense) of the parameters in (B13) and it generates higher dimensional operators in Q as well. A detailed analysis will be reported elsewhere.²⁷

REFERENCES

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† The invariance of S_F can be easily understood by first writing

$$S_F[WQW^{-1}] = \frac{1}{2}z\frac{\pi}{\beta}\sum'_{\alpha n}\text{Tr}[W^{-1}I_n^\alpha W, Q][W^{-1}I_{-n}^\alpha W, Q] \quad (\text{B14})$$

and then splitting

$$W^{-1}I_n^\alpha W = I_n^\alpha + W^{-1}[I_n^\alpha, W]. \quad (\text{B15})$$

The second term on the r.h.s. has nonzero matrix elements only in the upper left-hand and lower right-hand corners, i.e. in a 'small' neighborhood of the extreme diagonal components $(-N'_{\max}, -N'_{\max})$ and $(N'_{\max}-1, N'_{\max}-1)$, as can be seen by expanding the exponential form of W in powers of the I-matrices. Therefore, the second term in (B15) commutes with Q and we have $S_F[WQW^{-1}] = S_F[Q]$.

- ¹ For a review, see *The Quantum Hall Effect*, edited by R.E. Prange and S.M. Girvin (Springer-Verlag, Berlin, 1987)
- ² H. Levine, S. Libby and A.M.M. Pruisken, *Phys. Rev. Lett.* **51**, 20 (1983)
- ³ A.M.M. Pruisken, *Nucl. Phys.* **B235** [FS11], 277 (1984)
- ⁴ See, e.g., A.M.M. Pruisken and H.P. Wei, *AIP Conf. Proc.* **286**, 159 (1994)
- ⁵ A.M.M. Pruisken, *Phys. Rev.* **B32**, 2636 (1985)
- ⁶ A.M.M. Pruisken, *Phys. Rev. Lett.* **61**, 1297 (1988)
- ⁷ H.P. Wei, D.C. Tsui, M.A. Palaanen and A.M.M. Pruisken, *Phys. Rev. Lett.* **61**, 1294 (1988)
- ⁸ R. Laughlin, *Phys. Rev. Lett.* **50**, 1395 (1983)
- ⁹ R.B. Laughlin, M.L. Cohen, J.M. Kosterlitz, H. Levine, S.B. Libby and A.M.M. Pruisken, *Phys. Rev.* **B32**, 1311 (1985)
- ¹⁰ A.M.M. Pruisken and M.A. Baranov, *Europhys. Lett.* **31**(9), 543 (1995)
- ¹¹ A.M. Finkelstein, *JETP Lett.* **37**, 517 (1983); *Soviet Phys. JETP* **59**, 212 (1984); *Physica* **B197**, 636 (1994)
- ¹² F. Wilczek, *Phys. Rev. Lett.* **48**, 1144 (1982)
- ¹³ A.L. Fetter, C.B. Hanna and R.L. Laughlin, *Phys. Rev.* **B39**, 9679 (1989); **43**, 309 (1991)
- ¹⁴ A. Lopez and E. Fradkin, *Phys. Rev.* **B44**, 5246 (1991); *Phys. Rev. Lett.* **69**, 2126 (1992)
- ¹⁵ S.C. Zhang, H. Hansson and S. Kivelson, *Phys. Rev. Lett.* **62**, 82 (1989); See S.C. Zhang, *Int. J. Mod. Phys.* **B6**, 25 (1992)
- ¹⁶ S. Kivelson, S.C. Zhang and D.H. Lee, *Phys. Rev.* **B46**, 2223 (1992)
- ¹⁷ M.H. Cohen and A.M.M. Pruisken, *AIP Conf. Ser.* **286**, 205 (1993); *Phys. Rev.* **B49**, 4593 (1994)
- ¹⁸ B.I. Halperin, P.A. Lee and N. Read, *Phys. Rev.* **B47**, 7312 (1993)
- ¹⁹ J.K. Jain, *Phys. Rev. Lett.* **63**, 199 (1989)
- ²⁰ This was one of the main results of Lopez and Fradkin¹⁴, who discarded the edge states.
- ²¹ X.G. Wen, *Phys. Rev.* **B41**, 12838 (1990)

- ²² A.M.M. Pruisken, B. Škorić and M.A. Baranov, "Handling gauge invariance in the theory of the quantum Hall effect III: The instanton vacuum and chiral edge physics", ITFA preprint 98-17, cond-mat/9807241, to appear in *Phys. Rev. B*
- ²³ A.M.M. Pruisken and K. Schoutens, *Phil. Mag.* **B76** 807 (1997)
- ²⁴ B. Škorić and A.M.M. Pruisken, "The fractional quantum Hall effect: Chern-Simons mapping, duality, Luttinger liquids and the instanton vacuum", cond-mat/9812437, accepted for publication in *Nucl. Phys. B*
- ²⁵ A similar phenomenon is well known to occur in ordinary Fermi liquid theory where large frequency sums or non-Fermi level quantities contribute to the effective Landau parameters.
- ²⁶ M.A. Baranov, A.M.M. Pruisken and B. Škorić, "Handling gauge invariance in the theory of the quantum Hall effect II: Perturbative results", ITFA preprint 97-48, cond-mat/9712323, to appear in *Phys. Rev. B*
- ²⁷ M.A. Baranov and A.M.M. Pruisken, to be published. A simplified version of this theory is given in Appendix B.
- ²⁸ D. Pines and P. Nozières, *The Theory of Quantum Liquids: Volume 1* (W.A. Benjamin, Inc., New York 1966)